Matthew K. Tam
Joint work with Dr. Fran Aragón and Laur. Prof. Jon Borwein

School of Mathematical and Physical Sciences
University of Newcastle, Australia

AMSSC, 15th–17th July 2013

With generous support from AustMS and AMSSC
In **Sudoku** the player fills entries of an incomplete Latin square subject to constraints. As a decision problem, it is **NP-complete**.

Some questions to ponder during this talk are:

- How can we solve large Sudoku puzzles? \((n^2 \times n^2\) instances)
- If any single entry is removed, how many distinct solutions can a puzzle have?
- Can such characteristics be used to better understand why algorithms work?
The Douglas–Rachford method (a projection algorithm) was originally introduced in connection with PDEs arising in heat conduction. Applied to convex problems, the methods have a strong theoretical foundation, and its behaviour well understood.

I will discuss recent applications of the Douglas–Rachford method to a number of NP-complete combinatorial optimisation problems which are far from convex. Despite a lack of sufficient theoretical justification, the method performs quite satisfactorily.
Let $S \subseteq \mathbb{R}^n$. Recall, $S$ is convex if

$$\lambda S + (1 - \lambda)S \in S, \quad \forall \lambda \in [0, 1].$$

The (nearest point) projection onto $S$ is the (set-valued) mapping,

$$P_S x := \arg\min_{s \in S} \|s - x\|.$$ 

The reflection w.r.t. $S$ is the (set-valued) mapping,

$$R_S := 2P_S - I.$$
A Variational Toolkit

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The Douglas–Rachford Scheme

Theorem (Douglas–Rachford, Lions–Mercier)

Suppose $A, B \subseteq \mathbb{R}^n$ are closed and convex with $A \cap B \neq \emptyset$. For any $x_0 \in \mathbb{R}^n$ define

$$x_{n+1} := Tx_n \text{ where } T := \frac{I + R_B R_A}{2}.$$ 

Then $(x_n)$ converges to a point $x$ such that $P_A x \in A \cap B$.

$$A = \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \quad B = \{x \in \mathbb{R}^n : \langle a, x \rangle = b\}.$$
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\[A = \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \quad B = \{x \in \mathbb{R}^n : \langle a, x \rangle = b\}.\]
Let $E = \{e^j : j = 1, \ldots, 9\} \subset \mathbb{R}^9$ be the standard unit vectors. Define the array $X = (X_{ijk}) \in \mathbb{R}^{9 \times 9 \times 9}$ by

$$X_{ijk} = \begin{cases} 1 & \text{if } ij\text{th entry is } k, \\ 0 & \text{otherwise.} \end{cases}$$
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The constraints are:

$C_1 = \{X : X_{ij} \in E\}$

$C_2 = \{X : X_{ik} \in E\}$

$C_3 = \{X : X_{jk} \in E\}$

$C_4 = \{X : \text{vec}(3 \times 3 \text{ submatrix}) \in E\}$

$C_5 = \{X : X \text{ matches original puzzle}\}$

A solution is any

$$X \in \bigcap_{i=1}^{5} C_i.$$
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$P_{C_1}, P_{C_2}, P_{C_3}, P_{C_4}$ are simple to compute since, for any $x \in \mathbb{R}^9$,

$$P_E x = \{ e_j : x_j = \max_{1 \leq i \leq 9} x_i \}.$$ 

$P_{C_5}$ is also simple and given by setting $A_{ijk} = 1$ if the incomplete puzzle has a $k$ in the $ij$th position.
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Reformulate as a two set feasibility problem in the product space:

$$x \in \bigcap_{i=1}^5 C_i \subseteq \mathbb{R}^{9 \times 9 \times 9} \iff (x, x, x, x, x) \in D \cap C \subseteq (\mathbb{R}^{9 \times 9 \times 9})^5,$$

where

$$D := \{(x, x, x, x, x) \in (\mathbb{R}^{9 \times 9 \times 9})^5 : x \in \mathbb{R}^{9 \times 9 \times 9}\}, \quad C := \prod_{i=1}^5 C_i.$$
We tested the Douglas–Rachford method (C++) on various large suites of Sudoku puzzles. We give details of the implementation.

- **Initialise:** $x_0 = (x_0, x_0, x_0, x_0, x_0)$ for random $x_0 \in [0, 1]^{9 \times 9 \times 9}$.
- **Iterate:** By setting
  \[
  x_{n+1} = T x_n = \frac{x_n + R_C R_D x_n}{2}.
  \]
- **Terminate:** Either, if a solution is found, or if 10000 iterations have been performed. Specifically, a solution is found if
  \[
  P_D x_n \in C \cap D.
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- **Terminate:** Either, if a solution is found, or if 10000 iterations have been performed. Specifically, a solution is found if
  \[ \text{round}(P_D x_n) \in C \cap D. \]
Computational Results: Success Rate

Table 1. % Solved by Test Library.

<table>
<thead>
<tr>
<th></th>
<th>top95</th>
<th>reglib-1.3</th>
<th>minimal1000</th>
<th>ksudoku16</th>
<th>ksudoku25</th>
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<tbody>
<tr>
<td>DR</td>
<td>86.53</td>
<td>99.35</td>
<td>99.59</td>
<td>92</td>
<td>100</td>
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Douglas–Rachford for Combinatorial Optimisation
This ‘nasty’ Sudoku cannot be solved reliably (20.2% success rate) by the Douglas–Rachford method.

```
7 9 5
1 3
2 3 7
4 5 7
8

9 1
8 6
5 4 7
```
Computational Example: A ‘Nasty’ Sudoku

This ‘nasty’ Sudoku cannot be solved reliably (20.2% success rate) by the Douglas–Rachford method.

Success rate when any single entry is removed:
- Top left 7 = 24%
- Any other entry = 99%

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Success rate when any single entry is removed:
- Top left 7 = 24%
- Any other entry = 99%

Number of solutions when any single entry is removed:
- Top left 7 = 5
- Any other entry = 200–3800
Computational Results: Performance Comparison

Table 2.

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Douglas–Rachford for Combinatorial Optimisation
Computational Results: Performance Comparison

Table 2. Average Runtime (seconds).

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- Boolean satisfiability, protein folding, graph colouring.
- TetraVex, generalised 8-queens problem.
- Nonograms – a Japanese number painting.
- Matrix completion. e.g. low rank, various Hadamard matrices.
- Any suggestions?
Concluding Remarks

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Many resources can be found at the companion website:

http://carma.newcastle.edu.au/DRmethods/comb-opt/
The ‘Nasty’ Sudoku and its Unique Solution

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